THE PARIS REPORT ON CALCULUS IN SECONDARY SCHOOLS.

L'Enseignement for July–September contains the complete reports (referred to in the September MONTHLY) made at the Paris meeting of the International Commission on the Teaching of Mathematics. One of the two subjects there discussed, viz., the results of the introduction of the elementary notions of the calculus into secondary instruction, seems to be of such importance to the readers of the MONTHLY as to justify printing the following translation of the official summary of the general report made by Professor Beke, of Budapest, covering the replies made to the questionnaire. (For greater brevity, the differential and integral calculus will be referred to simply as the calculus.)

"1. *The place of the calculus in the secondary curriculum.*—In all countries where during the last twelve years a new plan of studies for secondary schools has been vigorously begun, a definite place of greater or less importance has been reserved for the notion of function, and as well, with very few exceptions, for the first elements of the calculus.[1]

"*A*. The elements of the infinitesimal calculus figure in the official programs of the schools, or in the plan of study established by the schools themselves, in the following countries: Bavaria, Wurtemberg, Baden, Hamburg, Austria, Denmark, France, Great Britain, Italy, Roumania, Russia, Sweden and Switzerland.

"*B*. They do not figure in the plan of study, but are elective in a large number of schools: Prussia, Saxony, Hungary, Australia; and they probably will be so before long in Holland, Norway, Belgium and Servia.

"2. *The range given to the calculus.*

"(*a*) It is nearly everywhere applied only to functions of one variable.

"(*b*) Instruction is everywhere given in the differentiation of polynomials, of rational functions (or at least quotients of two linear polynomials), and in most countries that of exponential and trigonometric functions and their inverses.

"(*c*) In most countries the notation of Lagrange is preferred to that of Leibnitz.

"(*d*) In most countries the notion of integral or primitive function is introduced. Everywhere the notion of integral follows that of derivative (in Bohemia these are treated simultaneously). In some countries the definite integral precedes the indefinite, but in most the reverse order is followed.

"3. *Applications of the infinitesimal calculus.*

"(*a*) Taylor's series figures in few programs. It is nevertheless taught in schools where the plans of study have for a long while included infinite series; in such cases the series for e^x, a^x, $\sin x$, $\cos x$, $\log(1 + x)$, $(1 + x)^m$, $\text{arc tan } x$ are developed. Professor Beke thinks that the treatment of Taylor's series is not yet sufficiently adapted to the secondary school.

"(*b*) The calculus is applied everywhere to investigations of maxima and minima.

"(*c*) It is also applied to physics, at least for the purpose of defining velocity

[1] On the status in America see remarks by Professors Smith and Van Vleck quoted on page 326.

and acceleration, but sometimes it finds a more extended application (centers of gravity, moments of inertia, potential, etc.). In Russia only elementary mathematics is used in physics.

" (*d*) The calculus is applied in geometry to the determination of areas and volumes, and it is here that the new method renders its greatest service on the economical side. But the old methods continue to be applied, especially Cavalieri's principle.

" 4. *The question of rigor.*—This is one of the most delicate points. From the side of the higher curriculum it is said that secondary teaching does more harm than good unless it adopts the rigorous method of a scientific treatment; on the other hand, the representatives of secondary instruction assert that the intellectual power of the average pupils does not permit a rigorous treatment of the calculus. Professors in secondary schools need to know the infinitesimal calculus in its modern rigorous form, but in their teaching they need to use an intuitive method, geometrical and mechanical considerations, and to progress gradually toward the necessary abstractions. This is also the surest way to arouse in the pupils the desire for rigor.

" (*a*) Irrational numbers are introduced almost everywhere incidentally in connection with the extraction of roots; the general theory is treated only in exceptional cases.

" (*b*) The notion of limit is introduced everywhere, nowhere do we rely on intuition alone. The elementary theorems relative to limits are adopted almost everywhere without explanation.

" (*c*) No allusion is made to continuous functions which nowhere admit derivatives. In certain schools instructors confine themselves to saying that at certain points the derivative may cease to exist.

" (*d*) In most schools the differential is not introduced; there is a good deal of confusion in the treatment of the notion of differential. It is greatly to be desired that the metaphysical fog of the infinitely small should not enter the secondary teaching.

" 5. *Fusion of the calculus with the subject matter of the secondary school.*—The new subjects should not be given as supplementary matter alongside of the old subject matter, but a complete fusion will have to be brought about between the two. The enlargement of the rôle of the notion of function and the introduction of the calculus can be successful only if the old program be reduced and made more economical. There will then result a relief, thanks to the fusion of the new material with the old, and the suppression of old out-of-date subject matter.

" 6. *The reform movement and the general opinion of educators.*—The very definite character of the results of our movement may be assured by its success and by the general opinion (l'opinion publique), always alert, of the representatives of education. The movement has gained everywhere the approval of the professors engaged in secondary education, but those engaged in higher education, who look at this from their own special point of view, do not always sympathize with our tendencies.

" We hear the complaint that a course in calculus is not followed with interest

by those who already have some knowledge of it. It is not difficult to refute this assertion. It suffices us to recall the favorable opinions which we have found among university professors in all countries, who look at our movement from a higher point of view."

To this should be added here the following extracts concerning the report of Professor D. E. Smith, the supplementary remarks made by Professor E. B. Van Vleck, and the British report made by Professor C. Godfrey.

"Professor D. E. Smith, the zealous reformer and one of the founders of the Commission, informs us, that the calculus does not figure in the secondary curriculum in the United States; it cannot even be made elective, since the pupils of the upper classes are very much absorbed in the preparation for college entrance examinations. So long as this (the calculus) is not put upon the program of these examinations, there is little chance that it will take a place in the subject matter of secondary teaching. Yet Professor Smith has the hope that before many years the calculus will be introduced in professional (technical) secondary schools. Knowing the great activity shown by our American colleagues, in the past and present, in regard to reform in mathematical teaching (we have only to recall the work of Professors D. E. Smith, Moore and Young) and seeing the immense scope of mathematical activity across the sea which dazzles our eyes and which is assuredly not without a favorable influence on the professors in secondary teaching, finally having confidence in the contact which in spite of distance exists between the workers of the two continents, we cannot doubt that before long the free development of mathematical teaching will have taken the decisive step."

Professor E. B. Van Vleck: "It may be added that to some degree the work of the first year or two of the American college course corresponds, in character, to that of the last year of the German gymnasium and the classes spéciales of the Lycées. The study of calculus is very commonly begun in the second year of the college course, and not infrequently it is taken by students in their first year. Furthermore, graphical representation for simple functions (linear and quadratic functions) has been increasingly introduced as a topic into the algebra of the high schools. From both of these facts it is clear that the tendencies now under discussion at this conference are also manifesting themselves visibly in the United States."

Professor C. Godfrey: "Broadly speaking the movement has received general support in England. Perhaps the most powerful stimulus is that of the engineers, as represented by Professor Perry. The physicists have long pressed for a modicum of calculus, and prefer to take it without too much mathematical rigor. The universities have progressively included more calculus in their examination papers for schools; these papers, together with those set by the Civil Service Commissioners (for admission to the army and the public service generally), are the most powerful lever that acts on the school curriculum. . . . Whatever opposition there has been to an introduction of the calculus at an early stage has come from those who fear that a diminished emphasis on the manipulative and formal side of algebra will have a bad effect. The question raised is this: What

algebraic equipment constitutes a firm basis for a superstructure of calculus? This is the only articulate objection that has found a voice. But the main obstacle is that most powerful force in educational matters—*vis inertiæ.*"

SUR UN PARADOXE ALGÉBRIQUE APPARENT

par G. LORIA, Université de Gênes, Italie

La contradiction signalée par M. Coolidge aux pages 184–5 de THE AMERICAN MATHEMATICAL MONTHLY de cette anneé peut se faire disparaitre de la manière suivante: Ecrivons l'equation

$$\begin{vmatrix} a & -b & c & -d \\ b & a & d & c \\ a' & -b' & c' & -d' \\ b' & a' & d' & c' \end{vmatrix} = 0$$

comme il suit

$$-\begin{vmatrix} a & -b & c & -d \\ a' & -b' & c' & -d' \\ b & a & d & c \\ b' & a' & d' & c' \end{vmatrix} = 0.$$

Appliquons a présent le théorème de Laplace sur le développement d'un déterminant et nous obtiendrons:

$$(ac' - a'c)^2 + (bd' - b'd)^2 + (ad' - a'd)^2 + (bc' - b'c)^2 - 2(a'b - ab')(c'd - cd') = 0.$$

ou bien, par des transformations algébriques faciles,

$$[(ac' - a'c) - (bd' - b'd)]^2 + [(ad' + bc') - (a'd + b'c)]^2 = 0.$$

Comme a, b, . . . sont toutes des quantités réelles cette équation unique se décompose dans les deux suivantes,

$$(ac' - a'c) = (bd' - b'd),$$

$$(ad' + bc') = (a'd + b'c),$$

qui sont précisément celles qu'a trouvees l'éminent professeur de l'Harvard University.

GÊNES, 28, VII, 1914.